# Schrödinger Networks

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**Abstract**

In this paper we discuss a discrete time independent Schroedinger equation defined on graphs. We show under appropriate hypotheses that Dirichlet, Neumann and Robin problems all have solutions. We then consider the inverse problem in recovering a potential function defined on the vertices of the graph from the Dirichlet to Neumann map. We show that we can always do this in critical circular planar graphs.

## 1 Preliminaries

A **Schroedinger Network** is the ordered 3-tuple $\Gamma = (G, K, q)$ where $G$ is a graph partitioned into boundary and interior vertices, $\partial G$ and $\operatorname{int} G$ respectively, $K$ is a Kirchoff matrix of conductivities as per usual, and $q$ is a vertex valued function, which we will call the **Schroedinger Potential Function**. We will always assume that $G$ is connected. Throughout the paper we will only consider positive conductivities and nonnegative $q$, though considering some of the recent work by Will Johnson, we could probably get away with a significantly larger set of conductivities and Schroedinger functions. Throughout the paper, we will consider $q$ to be both a $|V| \times |V|$ diagonal matrix with the same vertex ordering as $K$ and a function in $\mathbb{R}^V$. Similarly, we will not make the distinction between vectors in $\mathbb{R}^{|V|}$ and functions in $\mathbb{R}^V$, and hopefully the distinction presents no confusion.

Given $\Gamma$ and a function $\Phi \in \mathbb{R}^V$ we define the current at vertex $v$ to be

$$I_v(\Phi) = \sum_{u \sim v} (\Phi(v) - \Phi(u))\gamma_{uv} + q(v)\Phi(v)$$

where $\gamma_{uv}$ is the conductivity along the edge between vertices $u$ and $v$. If $I_\Phi \in \mathbb{R}^V$ is the function that maps $v$ to $I_v(\Phi)$, we clearly have $I_\Phi = (K + q)\Phi$ where the product is under normal matrix multiplication. We call a function $\Phi$ to be $\gamma q$**-harmonic** if it satisfies $[(K + q)\Phi]|_{\operatorname{int} G} = 0$.

## 2 Forward Problems

Here we will discuss existence and uniqueness results for the the Dirichlet, Neumann and Robin problems.

## 2.1 Dirichlet Problem

Given a Schroediner Network $\Gamma$ and a function $\phi \in \mathbb{R}^{\partial G}$, we wish to find a function $\Phi \in \mathbb{R}^{V}$ such that

$$\begin{aligned} I_{\Phi}|_{\operatorname{int} G} &= 0 \\ \Phi|_{\partial G} &= \phi. \end{aligned}$$

We begin with a lemma on convex functions.

**Lemma 2.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ convex function. The set of points such that $f' = 0$ is convex and corresponds to the global minimum of $f$. If $f$ is strictly convex the set where $f' = 0$ consists of a single point.*

*Proof.* This follows since the derivative of a weakly convex (resp. strongly convex) function is weakly increasing (resp. strongly convex). The details are left to the reader. □

**Lemma 2.2.** *If $F : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ (nonstrictly) convex function, then the set of points such that $\nabla F = 0$ is convex and corresponds exactly to the points that globally minimize $F$. If $F$ is strictly convex then $\nabla F = 0$ at at most one point.*

*Proof.* The claim follows from the previous lemma by restricting $F$ to appropriate line segments. The details are left to the reader. □

**Theorem 2.3.** *If $\Gamma$ is a Schroedinger network such that $q \geq 0$ then the Dirichlet problem has a unique solution.*

*Proof.* If $q = 0$, then the problem is the ordinary electrical network case, which we already know has a unique solution. So suppose $q$ has at least one component which is nonzero.

We will first show that $(K + q)$ is positive definite. We recall from [3] that

$$\mathbf{x}^T K \mathbf{x} = \frac{1}{2} \sum_{u,v \in V} (\mathbf{x}(u) - \mathbf{x}(v))^2 \gamma_{uv}$$

and hence

$$\mathbf{x}^T (K + q)\mathbf{x} = \frac{1}{2} \sum_{u,v \in V} (\mathbf{x}(u) - \mathbf{x}(v))^2 \gamma_{uv} + \sum_{u \in V} \mathbf{x}(u)^2 q(u) \tag{1}$$

is always nonnegative, and the term corresponding to $\mathbf{x}^T K \mathbf{x}$ is positive unless $\mathbf{x}$ is constant, in which case $\mathbf{x}^T q \mathbf{x}$ will be positive unless $\mathbf{x} = 0$. Hence $(K + q)$ is positive definite.

By computation, we observe that the Hessian of $\mathbf{x}^T (K + q)\mathbf{x}$ is $2(K + q)$, which is positive definite, and hence $\mathbf{x}^T (K + q)\mathbf{x}$ is a strictly convex function on $\mathbb{R}^{|V|}$.

Now fix a $\phi \in \mathbb{R}^{\partial V}$ and define the function $P \in \mathbb{R}^{\operatorname{int} V}$ as

$$P(\mathbf{y}) = \begin{pmatrix} \phi & \mathbf{y} \end{pmatrix} (K + q) \begin{pmatrix} \phi \\ \mathbf{y} \end{pmatrix}.$$

Clearly $P$ is a strictly convex function on $\mathbb{R}^{IntV}$. Since $K+q$ is symmetric and invertible, it permits an eigenvalue decomposition as

$$K + q = QLQ^t$$

where $L$ is a diagonal matrix of eigenvalues and $Q$ is an orthogonal matrix. Hence

$$P(\mathbf{y}) = \begin{pmatrix}\phi & \mathbf{y}\end{pmatrix} QLQ^t \begin{pmatrix}\phi \\ \mathbf{y}\end{pmatrix}.$$

We claim that $P(\mathbf{y}) \to \infty$ as $\|\mathbf{y}\| \to \infty$. This is clear since all of the entries in $L$ are positive since $K+q$ is positive definite (and hence has all positive eigenvalues) and $Q$ is orthonormal so

$$P(\mathbf{y}) \geq \lambda_{min}\|(\phi, \mathbf{y})\|^2_{\mathbb{R}^{|V|}} \geq \lambda_{min}\|y\|^2_{\mathbb{R}^{|\mathrm{int}\,G|}} \to \infty.$$

Applying Theorem 2.83 of [1] shows that $P$ attains an absolute minimum. Since $P$ is strictly convex, by Lemma 2, this implies that there is exactly one point $\mathbf{y_0}$ such that $\nabla P(\mathbf{y_0}) = 0$. In particular this implies that each of $\partial P/\partial y_i(\mathbf{y}_0) = 0$ where $y_i$ corresponds to any of the components of $\mathbf{y}$ on the interior of $G$. Differentiating the expression in equation (1), we see this is equivalent to

$$I_{(\phi,\mathbf{y}_0)}|_{\mathrm{int}\,G} = 0.$$

Hence $(\phi, \mathbf{y}_0)$ is the solution to the Dirichlet Problem.

□

At this point we note that our above work has shown that there is a well defined map $\Lambda_q : \mathbb{R}^{\partial G} \to \mathbb{R}^{\partial G}$ which maps function $\phi \in \mathbb{R}^{\partial G}$ to $[(K+q)\Phi]|_{\partial G}$ where $\Phi$ is the unique extension of $\phi$ shown in the Dirichlet problem. We leave it to the reader to verify that this map is linear. We can compute $\Lambda$ exactly as we did in electrical networks, namely by taking the Schur complement. If we order vertices in $G$ so that boundary vertices come before interior vertices, we can put $K$ into the block stricture

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where $A$ corresponds to boundary to boundary edges, $B$ corresponds to boundary to interior connections and $C$ corresponds to interior to interior connects. We order the entries of $q$ in the same way, and say that $q_{\mathrm{int}}$ is the square submatrix of $q$ corresponding to the interior vertices and $q_\partial$ is the square submatrix of $q$ corresponding to the boundary nodes. By exactly the same reasoning as in [3], we have that

$$\Lambda_q = (A + q_\partial) - B(C + q_{\mathrm{int}})^{-1}B^T.$$

We leave it to the reader to verify that since the Dirichlet problem has a a unique solution that $(C + q_{\mathrm{int}})$ must be invertible.

In the following section, we will show that $\Lambda$ is invertible. We call $\Lambda$ the Dirchlet to Neumann map, and $\Lambda^{-1}$ the Neumann to Dirichlet map.

## 2.2 Neumann Problem

**Theorem 2.4.** *Let $\Gamma$ be a Schroedinger network such that $q \geq 0$ and $q \not\equiv 0$. Let $\psi \in \mathbb{R}^{\partial G}$. Then there exists a unique $\Phi \in \mathbb{R}^V$ such that $(K+q)\Phi = \psi$ on $\partial G$.*

*Proof.* The proof is immediate, since $K+q$ is positive definite, and hence invertible. In particular, set

$$\Phi = (K+q)^{-1} \begin{pmatrix} \psi \\ 0 \end{pmatrix}.$$

□

## 2.3 Robin Problem

**Theorem 2.5.** *Let $\Gamma$ be a Schroedinger network such that $q \geq 0$. Let $\phi \in \mathbb{R}^{\partial G}$ and let $a \in \mathbb{R}^{\partial G}$ be such that $a > 0$ (pointwise). Then there exists a unique $\Phi \in \mathbb{R}^V$ such that $(K+q)\Phi = 0$ on $\operatorname{int} G$ and $(K+a)\Phi = \phi$ on $\partial G$ (where in the latter equation we are treating the function $a$ as a diagonal matrix).*

*Proof.* Since the above statement doesn't depend on the values of $q$ in $\partial G$ except that they are nonnegative, we will assume WLOG that $q|_{\partial G} = 0$.

If now $q$ is uniformly 0, then the problem reduces to the Neumann problem by replacing the boundary values of $q$ with $a$. So from here on suppose that $q(v) > 0$ for some $v \in \operatorname{int} G$.

Let $H$ be the vector space space of functions in $\mathbb{R}^V$ such that $[(K+q)\Phi]|_{\operatorname{int} G} = 0$. By existence and uniqueness to the Dirichlet problem, we know that $\dim H = |\partial G|$. Thus it is sufficient to show the linear map $M\Phi = [(K+a)\Phi]|_{\partial G}$ has trivial kernel on $H$. Suppose $(K+a)\Phi = 0$ on $\partial G$ and $(K+q)\Phi = 0$ on $\operatorname{int} G$. Recalling that $(K+q)$ is positive definite shows that if $\Phi \neq 0$, then

$$0 < \Phi^T (K+q)\Phi = \sum_{v \in \partial G} \Phi_v [(K+q)\Phi]_v + \sum_{v \in \operatorname{int} G} \Phi_v [(K+q)\Phi]_v.$$

Since $(K+q)\Phi = 0$ on $\operatorname{int} G$ and $q = 0$ on $\partial G$ and $K\Phi = -a\Phi$ on $\partial G$, we have

$$\Phi^T (K+q)\Phi = \sum_{v \in \partial G} -a_v \Phi_v^2$$

and if each $a_v > 0$ then we must have $\sum_{v \in \partial G} -a_v \Phi_v^2 \leq 0$, a contradiction. Hence the map $M$ is bijective from $H$ to $\mathbb{R}^{\partial G}$ and the Robin problem has a solution which is unique. □

# 3 The Augmented Graph

The reader should observe that if $v \in G$, then we trivially have

$$\begin{aligned} I_v(\Phi) &= \sum_{u \sim v} (\Phi(v) - \Phi(u))\gamma_{uv} + q(v)\Phi(v) \\ &= \sum_{u \sim v} (\Phi(v) - \Phi(u))\gamma_{uv} + q(v)(\Phi(v) - 0). \end{aligned} \tag{2}$$

This trivial modification gives has a useful interpretation, namely that by adding an additional boundary vertex to our original graph that has constant voltage 0, we can transform our Schroedinger potential function to edge conductivities. More precisely, given a Schroedinger network $\Gamma$, we define the **augmented graph** $\widetilde{G}$ to be the graph taken by adding a single boundary vertex $v_0$ which is connected to every other vertex with positive $q$. To form the **augmented electrical network** $\widetilde{\Gamma}$ we say $\widetilde{\gamma}_{uv} = \gamma_{uv}$ if $uv \in G$ and $\gamma_{uv_0} = q_u$ if $v_0$ is the new vertex. The diagram below should make things clear.

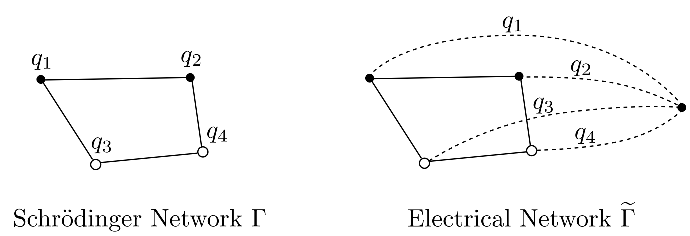


Figure 1: The augmented graph.

**Claim 3.1.** *A function $\Phi$ is $\gamma q-$harmonic on $\Gamma$ iff the function $\widetilde{\Phi}$ defined by $\widetilde{\Phi}|_G = \Phi$ and $\widetilde{\Phi}(v_0) = 0$ (where $v_0$ is the new vertex) is $\gamma-$harmonic on $\widetilde{\Gamma}$.*

*Proof.* The proof follows trivially from (2). □

**Remark 3.2.** *The* ***Maximum Principle*** *for electrical networks holds on the augmented graph.*

We also observe another way to compute $\Lambda_q$. If we order the boundary vertices so that the added vertex is the first vertex, then if $\widetilde{\Lambda}$ is the response matrix for $\widetilde{\Gamma}$ then $\Lambda_q$ is just obtained by removing the first row and column from $\widetilde{\Lambda}$. In particular, $\Lambda_q$ is just a submatrix of the response matrix for $\widetilde{\Gamma}$. We can thus apply the determinant connection formula (see [3] since we will not give a complete exposition). The following theorem will be useful in recovery, but first we need some notation.

- $\mathcal{C}(v_i, v_j)$ is the set of paths between the two boundary nodes $v_i$ and $v_j$ which don't contain any boundary vertices other than $v_i$ or $v_j$.
- $E_\alpha$ is the set of edges in a path $\alpha \in \mathcal{C}(v_i, v_j)$.
- $J_\alpha$ is the set of interior nodes which are not the endpoints of any edge in $\alpha$.
- $D_\alpha = \det \widetilde{K}(J_\alpha, J_\alpha)$.

**Theorem 3.3.** *Suppose $v_1, \ldots, v_n$ is the ordering of the boundary vertices used to compute $\Lambda_q$ and let $v_i, v_j \in \partial G$. There is a path in $G$ from $v_i$ to $v_j$ which does not contain any boundary vertices iff $(\Lambda_q)_{ij} < 0$. If there is no such path then $(\Lambda_q)_{ij} = 0$.*

*Proof.* Let $\widetilde{K}$ denote the response matrix for the augmented electrical network. We will call the added vertex $v_0$ (and assume 0-based indexing on our augmented matrices, and 1-based on our regular matrices so as to not disrupt out counting). By the determinant connection formula

$$(\Lambda_q)_{ij} \cdot \det \widetilde{K}(I, I) = (-1)^1 \sum_{\alpha \in \mathcal{C}(v_i, v_j)} \prod_{e \in E_\alpha} (\gamma(e) \cdot D_\alpha).$$

We recall that all principle proper submatrices of a Kirchoff matrix are positive definite (strictly), and hence $\det \widetilde{K}(I, I) > 0$ and $D_\alpha > 0$ for all $\alpha \in \mathcal{C}(v_i, v_j)$. By assumption all of the conductivities in $\widetilde{\Gamma}$ are positive. Thus $\sum_{\alpha \in \mathcal{C}(v_i, v_j)} \prod_{e \in E_\alpha} (\gamma(e) \cdot D_\alpha)$ is positive iff $\mathcal{C}(v_i, v_j)$ is nonempty. The statement follows immediately. $\square$

# 4 Dual Derivative and Residue Calculus

Given a circular planar graph $G$, we can define the dual and medial graphs of $G$. We refer the reader to [3] for the appropriate definitions. We denote the dual graph as $G^\dagger$ and we call the original graph the "primal" graph. We should note that when we say '$G$ is circular planar', we are referring to a specific circular planar embedding, which will be fixed in all contexts. Suppose we have a function $\Phi \in \mathbb{R}^V$ defined on $G$, then we define it's **dual derivative** to be $\Phi'$, defined on directed edges in the dual graph. If $v_1$ and $v_2$ are adjacent in $G$, then if $v_1^\dagger, v_2^\dagger$ are the vertices in $G^\dagger$ so that $\overrightarrow{v_1^\dagger v_2^\dagger}$ corresponds to the counterclockwise rotation of $\overrightarrow{v_1 v_2}$ then we define $\Phi'$ by

$$\Phi'(\overrightarrow{v_1^\dagger v_2^\dagger}) \stackrel{\text{def}}{=} \gamma_{v_1 v_2}(\Phi(v_2) - \Phi(v_1)).$$

In standard electrical networks, if $\Phi$ is $\gamma-$harmonic we define a dual function $\Phi^\dagger$ on the vertices of an electrical network to satisfy $\Phi^\dagger(v_2^\dagger) - \Phi^\dagger(v_1^\dagger) = \gamma_{v_1 v_2}(\Phi(v_2) - \Phi(v_1))$ for all $v_1 \sim v_2$. However this only yields a well defined function when the currents at each node sum to zero (which corresponds to adding the desired differences in a loop in the dual graph and getting zero). Since we no longer have this feature in $\gamma q-$harmonic functions, we can no longer define a function like this. The problem is much the same as finding an antiderivative of a meromorphic function around a pole with nonzero residue. We deepen the similarity to residues in the following discussion.

**Definition.** *Let $P = \overrightarrow{e_1}\overrightarrow{e_2} \ldots \overrightarrow{e_n}$ be a path of directed edges in the dual graph (with a particular direction). Let $G$ be a directed edge function on the dual graph. Then we define*

$$\int_P G \, d\ell \stackrel{\text{def}}{=} \sum_{i=1}^{n} G(\overrightarrow{e_i}).$$

If $\Phi$ is $\gamma q$-harmonic and $v$ is a vertex in $G$, we define the **Residue** at $v$, denoted by $\operatorname{Res}[\Phi, v]$ to be $\Phi(v)q(v)$. If $C = \overrightarrow{v_1 v_2}\overrightarrow{v_2 v_3} \ldots \overrightarrow{v_{n-1} v_n}$ is a directed cycle in the dual graph, due to our embedding, $C$ can be thought of a curve in the plane. Thus we can also talk about clockwise and counterclockwise via the winding number. We say $C$ is **Jordan** if $C$ is traversed counterclockwise, is closed, i.e. $v_1 = v_n$, and doesn't intersect itself, i.e., $v_i \neq v_j$ for any $1 < i, j \leq n$. We note that if $C$ is a Jordan path, then by the Jordan Curve Theorem, $C$ will divide the vertices of $G$ into two regions. We leave it to the reader to show that exactly one of these regions will contain boundary vertices. We define the interior region to be the one not containing any boundary vertices and we will denote that region (as a collection of vertices) as $\operatorname{int} C$.

**Theorem 4.1** (Residue Theorem)**.** *Let $\Phi$ be $q\gamma$-harmonic and let $C$ be a Jordan path. Then*

$$\int_C \Phi' \, d\ell = \sum_{v \in \operatorname{int} C} \operatorname{Res}[\Phi, v].$$

*Proof.* We sketch the proof (and leave the details to the reader). We will divide the $\operatorname{int} C$ into the cellular regions corresponding to each primal vertex in $\operatorname{int} C$. If $C_v$ is the Jordan path containing exactly the vertex $v$, we compute to verify that

$$\int_{C_v} \Phi' \, d\ell = \operatorname{Res}[\Phi, v] = \Phi(v) q(v).$$

Furthermore, when we consider the sum

$$\sum_{v \in \operatorname{int} C} \operatorname{Res}[\Phi, v] = \sum_{v \in \operatorname{int} C} \int_{C_v} \Phi' \, d\ell,$$

we observe that each edge in the dual graph in the interior region bounded by $C$ will appear in exactly two integrals, but with opposite orientations, and hence will cancel. The only edges that won't cancel will be the ones occuring in only one of the $C_v$ paths, which corresponds to being in the path $C$. We note that the edges will appear with the appropriate orientations, so our theorem statement is true. ☐

# 5 Recovering $q$

The inverse problem for Schroedinger Networks is to see whether the the map $q \mapsto \Lambda_q$ on a graph with fixed conductivities is globally injective. From now on, we will only consider **critical circular planar** graphs (see [3] for a description). We will spend the remaining section proving the following theorem:

**Theorem 5.1.** *If $G$ is a critical circular planar graph, then the map $q \mapsto \Lambda_q$ is globally injective, i.e. we can recover $q$ from $\Lambda_q$.*

Much of the following material on medial graphs and extensions come from a paper by Will Johnson ([2]). In fact, Johnson's argument for recovering conductivities in CCP graphs will be the basis for our argument. We will assume that all graphs from here on out are critical circular planar (see [3] for the definition).

## 5.1 Rooted Sets and Extension Systems

The rest of this paper will deal with the harmonic extension of functions defined on portions of the medial graph. We will notate the edges of the dual graph by $E^\dagger$ and the cells of the medial graph by $M$. We will consider sets $X \subseteq E^\dagger \cup M$. We will always assume that if a set contains an edge in the dual graph, then it contains the two cells in the medial graph corresponding to the vertices of the edge.

**Definition.** *Suppose $X \subseteq E^\dagger \cup M$. We construct the cellular adjacency graph $A_X$ by setting creating a vertex corresponding to each cell in $X$, and connecting two vertices in $A_X$ iff the cells they correspond to in $X$ share an geodesic edge. We say that $X$ is* ***cellularly connected*** *if $A_X$ is connected as a graph.*

**Definition.** *Suppose $X \subseteq E^\dagger \cup M$ and $D \subseteq X$ is a connected subset of the boundary of the medial graph. We say that $X$ is* ***dually connected through*** *$D$ (or just connected through $D$) if every cell in $X$ corresponding to a vertex in the dual graph can be connected to a boundary cell in $D$ through a path of edges and vertices in the dual graph which are contained in $X$.*

**Definition.** *Suppose $X \subseteq E^\dagger \cup M$. We say $X$ is rooted when all of the following are true:*

1. *$X$ is cellularly connected,*
2. *$X$ is connected through a connected subset of the medial graph,*
3. *whenever $X$ contains an edge in the dual graph, it contains at least one of the two cells of the primal cells of the medial graph adjacent to it.*

To make the paper readable and not overly tedious, we will make a great use of pictorial notation for conditions like the above one. We will abandon the pictorial distinction between boundary vertices and interior vertices, instead notating vertices from the primal graph by solid circles ($\bullet$) and vertices in the dual graph by empty circles ($\circ$). We will shade cells in the medial graph in our diagram to notate that they are in $X$. We will draw a dashed line between vertices in the dual graph to notate that the edge may or may not be in $X$, and we will draw a solid line to indicate that the edge definitely is in $X$. We will use the following type of diagram often, viewing each picture as a statement of inclusion holding at every vertex in the medial graph:

Let $\mathcal{P}(Y)$ denote the powerset of a set $Y$ and let $V(M)$ be the vertices of the medial graph. We have the following definition:

**Definition.** *We define an* ***Extension System*** *$E$ to be a set of functions defined on $\mathcal{P}(E^\dagger \cup M) \times V(M)$ which map to $\mathcal{P}(E^\dagger \cup M)$ such that*

1. *$e(\varnothing, v) = \varnothing$ for all $e \in E$, $v \in V(M)$,*
2. *$X \subseteq e(X, v)$ for all $e \in E, v \in V(M)$,*
3. *$e(X, v) \subseteq e(Y, v)$ for all $e \in E, v \in V(M)$ whenever $X \subseteq Y$. (Monotonicity)*

A final condition we *could* consider would be that if $e \in E$ and $v \in V(M)$ then $e(e(X, v), v) = e(X, v)$. All of the extension systems we consider will satisfy this, but it might be useful for posterity to consider useful extension systems which don't satisfy this.

**Definition.** *We say that a set $X \subseteq E^{\dagger} \cup M$ is* ***closed*** *in $E$ if $e(X, v) = X$ for all $e \in E$ and $v \in V(M)$. By monotonicity, any intersection of closed sets is closed, so we can talk about the smallest closed set containing $X$, which we will denote as $\overline{X}$ or $Cl_E(X)$ if the extension system being used isn't clear.*

**Definition.** *We call two extension systems, $E$ and $F$,* ***equivalent*** *if $Cl_E(X) = Cl_F(X)$ for all $X \in \mathcal{P}(E^{\dagger} \cup M)$.*

## 5.2 Electrical Extension System

We now present the extension system that was implicitly used by Will Johnson in [2]. We could get by with fewer functions, but it will be useful in the future to have them as we do. Basically if we have three cells in the medial graph around a vertex, then we have functions to extend to the fourth, and if we have two cells corresponding to adjacent vertices in the dual graph, then we can extend to the edge connecting them. We present them pictorially, with the assumption that any of the functions applied to a vertex not of the form shown will be the identity map.

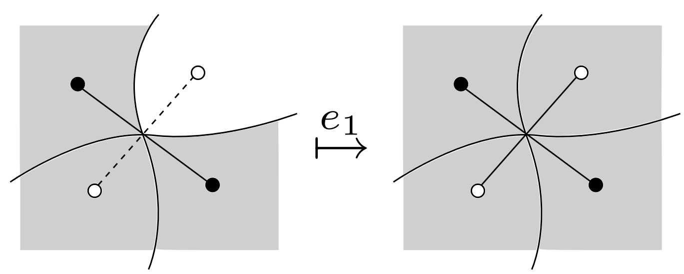


Figure 2: A pictorial representation for $e_1$.

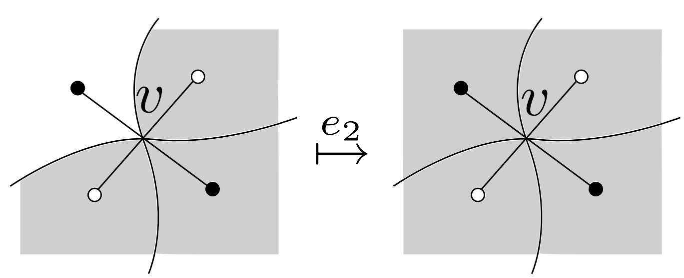


Figure 3: A pictorial representation for $e_2$.

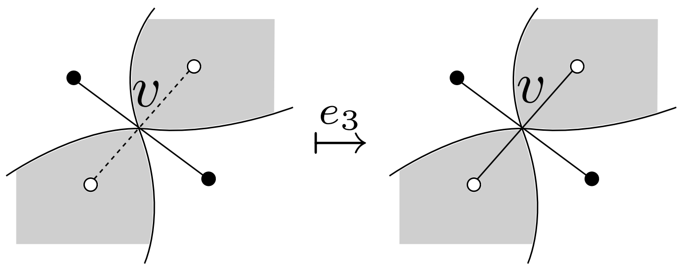


Figure 4: A pictorial representation for $e_3$.

We remark these functions clearly satisfy the requirements of an extension system.

### 5.3 Schrödinger Extension System

We now define another extension system, consisting of four functions $s_1, s_2, s_3$, and $s_4$. We will take the first two functions from the electrical networks. So let $s_1 = e_1$ and $s_2 = e_2$. Unfortunately, the third extension function for electrical networks turns out not to be what we want for Schroedinger networks (as we will later see), and hence we have to define two more functions to get around this.

We will define $s_3$ as follows (for the lazy readers, skip ahead to the pictures). Let $X \subseteq \mathcal{P}(E^\dagger \cup M)$ and $v$ is a vertex in the medial graph. Suppose both of the dual graph cells $d_1$ and $d_2$ that are adjacent to $v$ are in $X$ (though the edge between them potentially isn't). Now suppose there is a path of dual graph edges in $X$ which connects $d_1$ and $d_2$ such that for each edge in the cycle, the outer cell from primal graph which is adjacent to this edge is also in $X$ (we define outer to mean in the region containing boundary vertices). We also allow the case that the path from $d_1$ to $d_2$ is travels through a connected component of boundary cells (which all must be contained in $X$). In this case we define "outside" to mean the region not containing the boundary cells in the path. If the above conditions are satisfied, then $e_3$ gives adds everything inside of the cycle (though $e_3$ does not add the $d_1 d_2$ edge). We illustrate $s_3$ with a bunch of awesome pictures.

We now define $s_4$. Let $d_1, d_2 \in X$ be vertices adjacent to the vertex $v \in V(M)$ in the dual graph with an edge between them (which is not necessarily in $X$) and $d_1$ and $d_2$ are connected by a path of edges in the dual graph which are in $X$. If all of the cells in the primal graph cells in one of the regions bounded by the path connecting $d_1$ and $d_2$ then $e(X, v) = X \cup \{d_1 d_2\}$. As with $s_3$, we also include the possibility that the path between $d_1$ and $d_2$ consists partially (or entirely) of a connected path in the boundary of the medial graph which is contained in $X$. Again, we have pictures below.

We note that $s_1, s_2, s_3$ and $s_4$ preserve the rootedness of a graph. Before we continue, we need some definitions

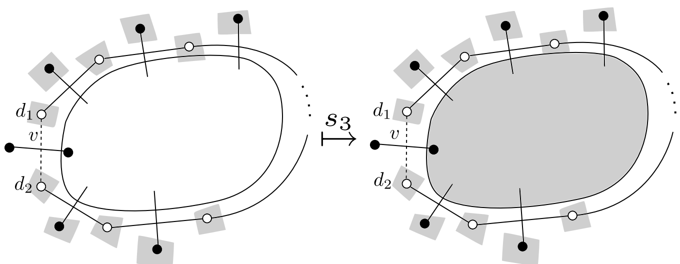


Figure 5: An example of $s_3$ where a loop does not contain a portion of the boundary.

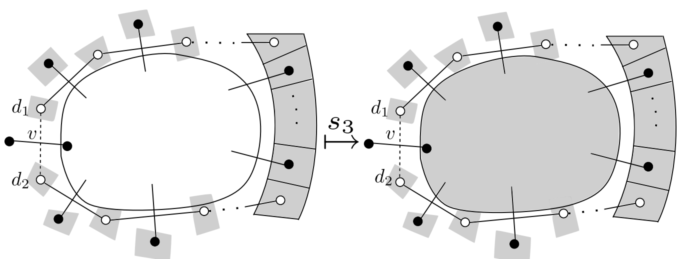


Figure 6: An example of $s_3$ where a loop *does* contain a portion of the boundary.

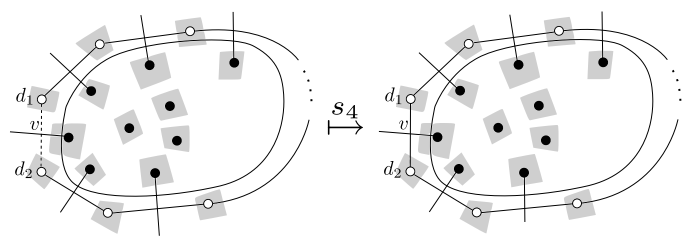


Figure 7: An example of $s_4$ where a loop does not contain a portion of the boundary.

## 5.4 Extensions of Rooted Sets

We wish to show that the electrical extension system, $\mathcal{E}$ and the Schroedinger extensions system, $\mathcal{S}$, are equivalent for rooted graph.

**Lemma 5.2.** *If $X$ consists of all of the boundary of the medial graph except possibly a single primal vertex, then $Cl_E(X) = E^\dagger \cup M$.*

This is very straightforward and the proof is left to the reader (see [2]).

**Corollary 5.3.** $s_3(X, v) \subseteq Cl_{\mathcal{E}}(X)$ *for all $v$ and $X$.*

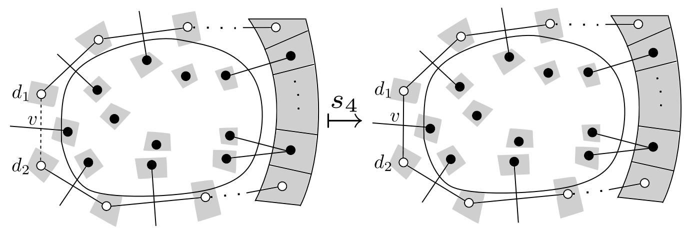


Figure 8: An example of $s_4$ where a loop *does* contain a portion of the boundary.

*Proof.* Apply the previous lemma to the subgraph contained in the loop. □

**Theorem 5.4.** *The Schroedinger and Electrical extension systems are equivalent for rooted graphs, i.e. if $X$ is rooted, then $Cl_{\mathcal{E}}(X) = Cl_{\mathcal{S}}(X)$.*

*Proof.* We first observe that if $s_1(X, v) = e_1(X, v) \subseteq Cl_{\mathcal{E}}(X)$ and similarly $s_2(X, v) = e_2(X, v) \subseteq Cl_{\mathcal{E}}(X)$. Furthermore, by the previous corollary, we have $s_3(X, v) \subseteq Cl_{\mathcal{E}}(X)$. Finally, we have $s_4(X, v) \subseteq e_3(X, v) \subseteq Cl_{\mathcal{E}}(X)$. Hence $Cl_{\mathcal{S}}(X) \subseteq Cl_{\mathcal{E}}(X)$ for any $X$.

Now suppose $X$ is rooted. We write $Cl_{\mathcal{E}}(X)$ as the last element in a finite sequence of sets $X_j$ such that $X_1 = X$ and $X_{j+1} = e_{i_j}(X_j, v_j)$. We will create a finite sequence $\{Y_j\}$ so that $X_j \subseteq Y_j$ for all $Y_j$ and $Y_j$ is rooted and $Y_j \subseteq Cl_{\mathcal{S}}(X)$. We define $Y_j$ recursively. Set $Y_1 = X_1 = X$. Suppose $Y_j$ has been defined as above for all $j \leq k$. If $i_k = 1, 2$ then we just set $Y_{k+1} = e_{i_k}(Y_j, v_j)$ which clearly satisfies our requirements since $e_1(Y_k, v_k) \subseteq Cl_{\mathcal{S}}(X)$ and $e_2(Y_k, v_k) \subseteq Cl_{\mathcal{S}}(X)$ and $e_1(Y_j, v)$ and $e_2(Y_j, v)$ are both also rooted. Now suppose $i_k = 3$. We want to show that there is a $Y_{k+1} \subseteq Cl_{\mathcal{S}}(X)$ such that $e_3(Y_k, v) \subseteq Y_{k+1}$ and $Y_{k+1}$ is rooted. This is actually not hard. The rootedness of $Y_k$ implies that if $d_1, d_2$ are dual vertices adjacent to $v$ which are both contained in $Y_k$, then each has a path to the boundary which is contained in $Y_k$. If these paths intersect at a dual cell $d_0$, we concatenate the path from $d_1$ to $d_0$ with the path from $d_0$ to $d_2$. Otherwise we have a path from $d_1$ to $d_2$ which intersects a connected component of the boundary which is contained in $Y_k$. Since $Y_k$ is rooted, at each edge along our path from $d_1$ to $d_2$, at least one of the adjacent primal cells will be contained in $Y_k$. We define $Y'$ as $s_2$ sequentially applied to each of the vertices along the dual edges on the path from $d_1$ to $d_2$. We then set $Y_{k+1} = e_4(e_3(Y', v), v)$ and observe that $\{d_1 d_2\} \in Y_{k+1}$ and $Y_{k+1}$ is rooted. Furthermore, $Y_{k+1}$ is clearly a subset of $Cl_{\mathcal{S}}(X)$ since we just applied a bunch of extension functions in $\mathcal{S}$ to $Y_k$, which was a subset of $Cl_{\mathcal{S}}(X)$. If $X_n = Cl_{\mathcal{E}}(X)$, then we have thus shown that $X_n = Cl_{\mathcal{E}}(X) \subseteq Y_n \subseteq Cl_{\mathcal{S}}(X)$, so $Cl_{\mathcal{E}}(X) \subseteq Cl_{\mathcal{S}}(X)$. □

We note this implies that the $\mathcal{E}$ closure of a rooted graph is rooted.

## 5.5 Consistent Extensions

In extending functions defined on a primal graph and the edges of the dual network, we will need an additional notion of extension which is slightly stronger than the above definition. Everything to follow relies heavily on work done in [2].

**Definition.** *Let $E$ be an extension system. Set $E_0 \subseteq E$, which we will call the **simple extension functions** of $E$ (at this point $E_0$ can be any fixed subset of $E$). Suppose $X \subseteq E^\dagger \cup M$. We say $X'$ is a **simple extension** if $X' = e(X, v)$ for some $e \in E_0$ and $v \in V(M)$. We say $X'$ is **consistent** if*

1. *$X' \setminus X$ has at most one edge in $E^\dagger$ and at most one cell in $M$.*
2. *If $x \in X' \setminus X$, then $x \notin e(X, v)$ for any $e \in E_0$ and $v \in V(M)$.*

*In general, if $X \subseteq X' \subseteq Cl_E(X)$, and $X'$ can be written as a sequence of applications of functions in $E$ to $X$, we will say that $X'$ is an $E$-extension of $X$, but if $X'$ can be written as a sequence of simple consistent extensions, then we say $X'$ is a **consistent** extension of $X$.*

The motivation for the previous definition is that if we know the values of a function on $X$ which satisfies rules corresponding in some way to an extension system (for instance being $\gamma q$-harmonic), then the values of an extension of our function on a consistent extension of $X$ will be uniquely determined and not contradictory, i.e., will still be $\gamma q$-harmonic.

We will now fix the simple extension functions with respect to the Schroedinger Extension system to be $\mathcal{E}_0 = \{e_1, e_2, e_4\}$.

**Definition.** *If $X \subseteq M \cup E^\dagger$, we define the number* $\operatorname{rank}(X)$ *to be the number cells of in $X$ minus the number of interior vertices in $X$ (an interior vertex is one where the four neighbouring cells in $M$ are in $X$).*

**Lemma 5.5.** *If $X'$ is an electrical extension of $X$, then* $\operatorname{rank}(X') \leq \operatorname{rank}(X)$.

*Proof.* This is follows immediately since $e_1$ and $e_2$ add a single cell and add at least one interior vertex. Furthermore, $e_3$ doesn't change rank at all. ☐

**Corollary 5.6.** *If $X'$ is a Schroedinger extension of $X$, then* $\operatorname{rank}(X') \leq \operatorname{rank}(X)$.

*Proof.* By the proof of Theorem 5.4, every Schroedginer extension is also an electrical extension. ☐

**Theorem 5.7** ([2])**.** *Let $X \subseteq M \cup E^\dagger$ be a connected set of cells. Then* $\operatorname{rank}(X)$ *is one more than the number of geodesics which pass through the interior of $X$, where we count a geodesic multiply if it exits and reenters $X$. In particular, if $X$ is convex, then* $\operatorname{rank}(X)$ *exactly equals the number of geodesics which pass through the interior of $X$ plus one.*

The proof can be found in [2].

**Lemma 5.8.** *If $s_3(X,v) \setminus X$ contains a cell in the medial graph which is not adjacent to $v$, then $\mathrm{rank}\,(s_3(X,v)) < \mathrm{rank}\,(X)$. If $s_3(X,v) \setminus X$ contains no cells which are not adjacent to $v$, then $s_3(X,v) = s_2(X,v')$ for some $v' \in V(M)$.*

*Proof.* Since the medial graph behaves well with boundary to boundary and boundary to interior contractions, it suffices to assume that $X$ contains all of the boundary cells of the medial except possibly for one primal cell, which we denote $p$, which is adjacent to $v$. Clearly every geodesic in the medial graph must pass through the interior of $X$, so $\mathrm{rank}\,(X) \geq G+1$ where $G$ is the total number of geodesics. Denote the two geodesics which reach the boundary while adjacent to $p$ by $g_1$ and $g_2$. Suppose there is a cell in interior of the medial graph which is not in $X$. Denote it by $c$.

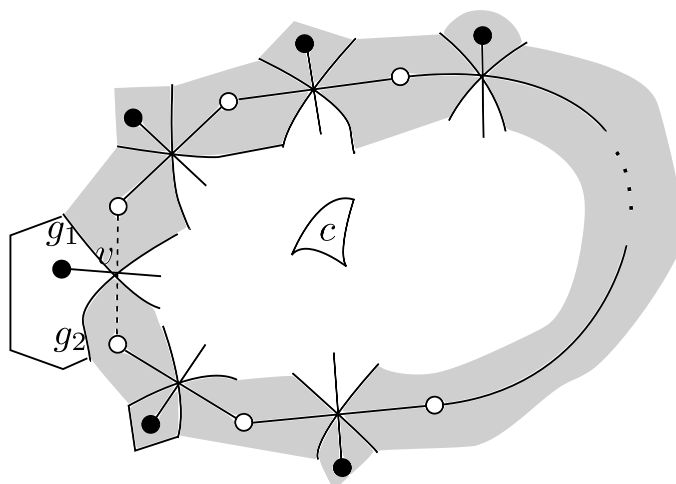


Figure 9:

Since $G$ is critical, $c$ must have be bounded by at least 3 geodesics (since otherwise we'd have a lens). At most two of these geodesics can be $g_1$ or $g_2$. In particular, the one that is neither $g_1$ nor $g_2$, must both leave and reenter the interior of $X$ (since all boundary cells other than $p$ are in $X$). Hence $\mathrm{rank}\,(X) \geq G+2$. Clearly $s_3(X,v) = Cl_{\mathcal{E}}(X) = M \cup E^{\dagger}$. Since $\mathrm{rank}\,(M \cup E^{\dagger}) = G+1$, we have $\mathrm{rank}\,(s_3(X,v)) = G+1 < \mathrm{rank}\,(X)$. $\square$

**Corollary 5.9.** *Suppose $X$ is rooted. If $\mathrm{rank}\,(X) = \mathrm{rank}\,(Cl_{\mathcal{E}}(X))$, then $Cl_{\mathcal{E}}(X) = Cl_{\mathcal{E}_0}(X)$*

*Proof.* It is sufficient to show that if $X$ is rooted and $\mathrm{rank}\,(X) = \mathrm{rank}\,(Cl_{\mathcal{E}}(X))$, then we can write $s_3(X,v)$ as a repeated composition of $X$ by $s_1, s_2$ and $s_4$. Suppose we have a path of dual edges which are contained in $X$, which would form a cycle if we added the edge $d_1 d_2$, where $d_1$ and $d_2$ are dual cells in $X$ which are adjacent to $v$. Suppose further that $X$ also contains all of the primal cells adjacent to edges in $P$ on the exterior of a region

$R$ determined by the cycle formed by $P$ and $d_1d_2$ (so that $s_3$ will add something to $X$). By the previous Lemma, we know that $X$ must contain all of the cells in $R$ except possibly the cell adjacent to $v$. Let $b$ denote the cell in the primal graph in the bounded region which is possibly not in $X$, and let $b'$ denote the primal cell which is connected to $b$ which is across the edge $d_1d_2$. If $b$ is a boundary cell, then $b \in X$ since $X$ is rooted. If $b$ is not a boundary cell, then $b$ must be connected to primal cell $b'' \neq b'$ since $G$ is critical. By the previous Lemma, the two dual cells $d_1'$ and $d_2'$ adjacent to the medial vertex between $b$ and $b''$ must be contained in $X$ since they are either in $R$ or on the path $P$. Since $X$ is rooted, we know that the subset of the dual graph which is in $X$ is connected to the boundary, and hence there are paths from $d_1'$ to $P$ and from $d_2'$ to $P$. Since no dual cells inside $R \setminus P$ are connected by an edge to any dual cell in $((M \cup E^\dagger) \setminus R) \setminus P$, we can assume these paths are completely contained in $R$. If these paths intersect in $R \setminus P$, then by combining these paths, we have a path $P'$ from $d_1'$ to $d_2'$ which doesn't contain any edges from $P$, and hence it will bound a subregion $R_0 \subseteq R$ which cannot contain $b$ since no edge in $P$ is contained in the path from $d_1'$ to $d_2'$ (and hence $d_1d_2$ cannot be in $P'$, and no subregion of $R$ can contain $d_1d_2$ but not $b$). Similarly, if the paths from $d_1'$ to $P$ and $d_2'$ to $P$ don't intersect, then we form a path from $d_1'$ to $d_2'$ by concatenating the path from $d_1'$ to $P$ with an appropriate segment along $P$ and then concatenating again with the path from $P$ to $d_2'$ (see picture below).

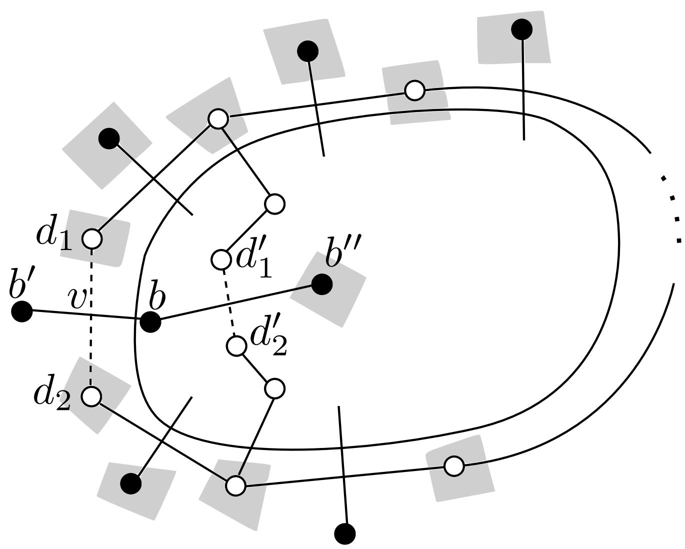


Figure 10:

This path, with the addition of $d_1'd_2'$ bounds a subregion $R_2 \subseteq R$, which cannot contain $b$ since $d_1d_2$ is not in the path we chose (the reader can fill in this implication), and hence by applying $s_4$, we see that a simple extension of $X$ contains the edge $d_1'd_2'$. By an appropriate application of $s_2$, we see that $b$ is in a simple extension of $X$. Clearly we can get the remaining dual edges in $R$ by applying $s_1$ appropriately within the region. $\square$

**Theorem 5.10.** *Let $X$ be a $\mathcal{E}$ closed, rooted set. Let $a$ be a boundary cell in the medial graph which is adjacent to a cell in $X$. If $Y$ is a simple extension of $X \cup \{a\}$ then $Y$ is a consistent extension. In particular, $Cl_{\mathcal{E}}(X \cup \{a\})$ is a simple consistent extension of $X \cup \{a\}$.*

*Proof.* We ovserve that $\operatorname{rank}(X \cup \{a\}) = \operatorname{rank}(X)+1$ because $X$ is closed. Furthermore, since the geodesic that separates $X$ and $a$ cannot be pass through the interior of $X$ (since it must correspond to one of the half planes defining $X$), and it does pass through the interior of $X \cup \{a\}$, we know that $\operatorname{rank}(X \cup \{a\}) > \operatorname{rank}(X)$. By Corollary 5.6 we know that $\operatorname{rank}(Cl_{\mathcal{E}}(X \cup \{a\}) \leq \operatorname{rank}(X \cup \{a\})$. Hence $\operatorname{rank}(X \cup \{a\}) = \operatorname{rank}(Cl_{\mathcal{E}}(X \cup \{a\})$. By Lemma 5.6, we thus know that if $Y$ is an extension of $X \cup \{a\}$ then $\operatorname{rank}(X \cup \{a\}) \geq \operatorname{rank}(Y) \geq \operatorname{rank}(Cl_{\mathcal{E}}(X \cup \{a\})$ and hence we have equality throughout.

We now will write an increasing sequence of sets corresponding to applying various functions in $\mathcal{E}_0$. Set $X_0 = X \cup \{a\}$ and $X_n = Y$ such that $X_{i+1} = e_{i_j}(X_i, v_i)$, and suppose WLOG that $X_{i-1} \subsetneq X_i$ for all $1 \leq i \leq n$. Since the rank doesn't decrease, we cannot have at any stage of our process a cell in the medial graph which would be added by two separate extension functions (or the same one applied at two different vertices) since then we would be adding exactly one cell and increasing the number of interior vertices by at least two (one for each way we could add the cell) and hence the rank would drop, which is a contradiction.

We now need only verify that no edge in the dual graph could be added in two different ways. A quick examination of our extension functions shows that the only way that we could have two different ways of adding a dual edge would be to have a vertex $v$ in the medial graph where all four adjacent cells were in $X_i$ but the dual edge was not, in which case either applying $s_1$ or $s_4$ to the vertex would add the dual edge. Let $i$ be the smallest $i$ such that $X_i$ contains at most three of the cells around $v$. We observe that such an $i$ exists since $X$ is closed so all vertices in the medial graph have exactly 0, 2, or 4 cells adjacent to them which are in $X$. Hence if a vertex is adjacent to four cells in $X \cup \{a\}$, they must have already been in $X$, and hence the dual edge would have also already been in $X$ since $X$ is closed. Since at each stage, we can add at most 1 cell in the dual graph, we know that there are exactly 3 cells around $v$ in $X_i$. We let $b$ be the cell which is in $X_{i+1} \setminus X_i$. The cell $b$ cannot be added by any function applied to the vertex $v$, since if so, we would be adding the dual edge. Thus we must be adding $b$ by applying either $s_1$ or $s_2$ to another vertex. In either case, $b$ must be adjacent to two anticorners (one at $v$ and one at the other vertex) and hence adding $b$ adds a single cell and two interior vertices, which would lower the rank, which is a contradiction.

The last claim follows since Corollary 5.9 shows that $Cl_{\mathcal{E}}(X \cup \{a\})$ is a simple extension of $X \cup \{a\}$ (and hence by what we have just proven, $Cl_{\mathcal{E}}(X \cup \{a\})$ is a simple consistent extension. $\square$

**Theorem 5.11.** *Let $X$ be rooted and closed in $\mathcal{E}$ and let $a \in M \setminus X$ be a boundary cell which is adjacent to $X$. If $Y$ is a simple extension of $X \cup \{a\}$, then $s_3(Y, v) \setminus Y$ consists of at most one element, which is a primal cell.*

*Proof.* By the previous theorem $Y$ is a simple consistent extension. The addition of any cell other than the primal cell adjacent to $v$ is impossible by Lemma 5.8. The addition of any dual edges is impossible by exactly the same argument as in the previous theorem (since $X$ is closed, we can show exactly as above that we would drop in rank somewhere in extending to $Y$). □

## 5.6 Extending Functions and Finding $q$

We now present a way of finding $q$ from $\Lambda_q$. Our general strategy will be to eliminate first delete all boundary to boundary edges, since they turn out to be insignificant. Next we'll contract boundary spikes by finding an appropriate functions on the medial graph which have value 1 on the boundary portion of a boundary spike, and 0 on the interior portion. We'll repeat this process until we exhaust our graph.

We recall two lemmas from [3]:

**Lemma 5.12.** *Every CCP graph has at least 3 boundary spikes or boundary to boundary edges.*

**Lemma 5.13.** *Contracting a boundary to boundary edge or boundary spike on a CCP graph yields a CCP graph (or possibly a disconnected graph with only CCP connected components).*

**Step 1.** We first observe that since the Schur Complement is linear in boundary to boundary connections, we can just subtract the already known conductivities from the appropriate components of the original response matrix to get the response matrix of the network with the boundary to boundary edge deleted.

**Step 2.** By Lemma 5.12, we know that after deleting all boundary to boundary edges, we are left with at least one boundary spike. We wish to find some sort of mixed boundary conditions that will always have a solution and will guarantee that the function on the interior will have value 0 at the interior node of the boundary spike, and value 1 on the boundary component. We observe that there is a geodesic $g$ which passes through the boundary spike. Pick all of the the boundary cells in the medial graph on the side of $g$ which doesn't contain the boundary vertex on the stated boundary spike. We construct a function (which will turn out to be $\gamma q$-harmonic), as follows. Define the set $X \subseteq M \cup E^\dagger$ to be all of the boundary medial cells on the side of $g$ which does not contain the chosen boundary vertex on the boundary spike. We remark that $X$ is rooted. Define the triple $F = (f, \phi, e)$ where $f$ is some function on primal vertices (a voltage function), $\phi$ is a function on boundary vertices (a boundary current function) and $e$ is a directed edge function on the dual graph (an interior current function). We proceed with several lemmas.

**Lemma 5.14.** *Let $\Gamma$ be a Schroedginer network (recall we assume that $G$ is connected) and suppose $v \in \partial G$ be a boundary vertex. Suppose* *that $\Phi_1$ and $\Phi_2$ are $\gamma q$-harmonic on $\Gamma$ and $\Phi_1|_{\partial G\setminus\{v\}} = \Phi_2|_{\partial G\setminus\{v\}}$ and $\Lambda_q(\Phi_1)|_{\partial G\setminus\{v\}} = \Lambda_q(\Phi_2)|_{\partial G\setminus\{v\}}$. Then $\Phi_1 = \Phi_2$.*

*Proof.* By linearity it's sufficient to show that if $\Phi$ is $\gamma q$-harmonic, $\Phi|_{\partial G\setminus\{v\}} = 0$ and $\Phi|_{\partial G\setminus\{v\}} = 0$ then $\Phi = 0$ on $G$. If $\Phi(v) = 0$, we are done, so assume otherwise. Without loss of generality, assume that $\Phi(v) = 1$. Order the boundary vertices so that $v$ is the first one (and say $v = v_1$). Since $G$ is connected, $v$ must be connected through the interior to at least one other boundary vertex $v_j$. By Lemma 3.3, we know that $(\Lambda_q)_{1j} < 0$. Since $(\Lambda_q\Phi)_j = (\Lambda_q)_{1j} < 0$, we have a contradiction, since we assumed that $\Lambda_q\Phi(v_j) = (\Lambda_q\Phi)_j = 0$. □

**Lemma 5.15.** *Suppose $F = (f, \phi, e)$ is defined on $M \cup E^\dagger$ and $f$ is $\gamma q$-harmonic, $\phi = \Lambda_q(f)$ and $e$ is the dual derivative of $f$. If $f = 0$ on all primal cells of $X$ (where $X$ is defined as above) and $\phi = 0$ on all primal cells $v$ in $X$ whenever the two dual boundary cells adjacent to $v$ are also in $X$, then $f$ and $e$ are all $0$ on $Cl_{\mathcal{E}}(X)$.*

*Proof.* Observe that $X$ is rooted. We observe that the values of $f$ and $e$ will always be uniquely determined on the cells and dual edges that are added by $s_1$ and $s_2$. Since no application of an extension function will add a boundary cell, and $\phi$ is specified on all of the primal dual cells in $Cl_{\mathcal{E}}(X)$, by the previous lemma $f$ and $e$ will be uniquely determined on any cells or dual edges added by $s_3$. Similarly, by the Residue Theorem, $f$ and $e$ will be uniquely determined on any cell or dual edge added by $s_4$. In particular, $f$ and $\phi$ are uniquely determined on all of $Cl_{\mathcal{E}}(X)$, which is just the entire region on the appropriate side of $g$. □

**Theorem 5.16.** *Let $X$ be rooted and closed in $\mathcal{E}$, and suppose $X$ contains at least two adjacent boundary cells of the medial graph. Suppose $F = (f, \phi, e)$ is a function on a subset of $X$ such that*

1. *$f$ is defined on all primal cells in $X$,*
2. *$e$ is defined on all dual edges in $X$,*
3. *$\phi$ is defined on a primal boundary cell $v$ iff the adjacent two boundary dual cells are also in $X$, and $\phi$ is the sum of the currents coming into $v$, all of which are defined,*
4. *$e$ satisfies the dual derivative condition at any vertex in the medial graph when all four adjacent cells are in $X$,*
5. *if $v \in \operatorname{int} G \cap X$, and all adjacent vertices of $v$ are also in $X$, then $f$ is $\gamma q$-harmonic at $v$.*

*If $a \in M \setminus X$ is a boundary cell in the medial graph adjacent to $X$, then there is a unique extension of $F$ to $Cl_{\mathcal{E}}(X \cup \{a\})$ which satisfies the above properties on $Cl_{\mathcal{E}}(X \cup \{a\})$.*

*Proof.* We observe that $a$ can be adjacent to at most one boundary cell in $X$, since otherwise due to the circular ordering of boundary cells in the medial graph, we would need the left and right neighbors of $a$ to be contained in $X$, but since $X$ is closed, $a$ would have to be separated by a geodesic. But no geodesic bounds a region with exactly one boundary cell since the graph is connected (so no such region with exactly one cell could exist) and critical (so we can't have a region bounded by a geodesic with interior cells and exactly one boundary cell, since then we'd have a lens). If $a$ is a primal cell, we specify the value of $f$ arbitrarily on $a$. If $a$ is a dual cell, we specify the value of $\phi$ arbitrarily on the boundary primal cell in $X$ which is adjacent to $a$ (by what we have said there is exactly one such cell in $X$. We note that by assumption (3), $\phi$ cannot already be defined on $a$. We also note that the assumption that $X$ has at least two boundary cells, defining $\phi$ on the cell adjacent to $a$ would not contradict (3).

We now will extend $F$ using our extension functions. By Theorem 5.10 there is a consistent extension (of cells and edges) from $X \cup \{a\}$ to $Cl_{\mathcal{E}}(X \cup \{a\})$. We will extend $F$ in the same order as we extend $X \cup \{a\}$ to $Cl_{\mathcal{E}}(X \cup \{a\})$. Let $Y$ be any simple consistent extension of $X \cup \{a\}$ such that $F$ has been extended onto $Y$ in such a manner that satisfies all of the above conditions.

Let $s_1(Y, v)$ be an extension of $Y$. By Theorem 5.10, we know that $s_1(Y, v)$ is a consistent extension of $Y$. We define $e$ on the dual edge which is added to satisfy the dual derivative condition, so (2) and (4) are satisfied. We add no primal cells so (1) is satisfied. We note that we can never complete a any simple cycle in $X$ (a cycle which contains a single face in the dual graph), since then we could instead apply $s_4$, contradicting the fact that our extension (of cells and edges) is consistent. Hence in our application of $s_1$, we are always satisfying 5. We now consider condition 3. Let $d$ denote the dual cell we add. If $d$ is adjacent to a primal boundary cell $p$ which was already adjacent to another dual boundary cell, say $d'$, then we will need to define $\phi$ on $p$. Let $d_0$ be the dual cell that is also adjacent to $v$ and which has a dual edge connected to $d$. Since the edge $d_0d$ is not in $Y$, but $Y$ is connected, we know that $d_0$ is dually connected to $d'$ (possibly through connected components of the boundary of the medial graph which are also in $Y$). In particular, there is a path from $d_0$ to $d'$, and when we add the edge $dd_0$ and the cells $d$, $p$ and $d'$, we get a closed loop (in the sense of $s_3$) which is in $s_1(Y, v)$, and which bounds a region $R \subseteq M \cup E^\dagger$ (by the Jordan Curve Theorem there will be two regions;pick $R$ to be the region containing the boundary cells $d, p$ and $d'$). We create the simple extension $Y'$ of $Y$ by applying $s_2$ at every medial vertex on a dual edge in $P$. By Theorem 5.10, we know that $Y'$ is a consistent simple extension of $X \cup \{a\}$, and hence by Theorem 5.11, we know that $s_3(Y', v) \setminus Y'$ cannot contain any dual edges. But $s_3(Y', v)$ must contain all of the dual edges corresponding to primal edges connected to $p$. Since $Y' \setminus s_1(Y, v)$ contains no dual edges, all of the dual edges crossing primal edges connected to $p$ must be in $s_1(Y, v)$ already, and hence $e$ must be defined on them. We thus define $\phi$ on $p$ as the appropriate sum of currents entering $v$.

We now consider applying $s_2$ to $Y$. We define $f$ on $s_2(Y, v)$ so that the dual derivative condition is satisfied (and since $s_2(Y, v)$ is a simple consistent extension of $Y$, the value of $f$ will be uniquely determined in this manner). Thus conditions (1), (2), (4), and (5) will be satisfied. As before, condition (4) requires some care. Call the primal cell that we add $p$. If adding $p$ is adjacent to two dual boundary cells in $Y$, we will need to specify $\phi$ at $p$. Let $d_1$ and $d_2$ be the adjacent boundary cells. Since our graph is dually connected (possibly through a connected component of the boundary), there is a path $P$ in $Y$ from $d_1$ to $d_2$. Adding $p$ would make this a loop. If $P \cup \{p\}$ contains the entire boundary, then by the uniqueness of the Dirichlet problem, the current at $p$ would be determined. If $P$ does not contain the entire boundary, it must contain some dual edge $\ell$, which contains some medial vertex, $v_0$. If we form the simple consistent extension $Y''$ of $s_2(Y, v)$ by applying $s_2$ along every dual edge in $P$, then $s_3(Y'', v_0)$ contains all of the dual edges in $R$, and in particular all of the dual edges crossing primal edges that have $p$ as an endpoint. By Theorem 5.11, $s_3(Y'', v_0) \setminus Y''$ contains no dual edges. Clearly $Y'' \setminus Y$ contains no dual edges. Hence $s_3(Y'', v_0) \setminus Y$ contains no dual edges, and hence $Y$ already contains all of the dual edges which cross primal edges that have $p$ as an endpoint. Hence $e$ is already defined on them, and we can uniquely define $\phi$ on $p$ as the sum of the entering currents.

We lastly need to consider the extension $s_4(Y, v)$. The function $s_4$ adds no cells so (1) and (3) are automatically satisfied on $s_4(Y, v)$. If $s_4(Y, v) \setminus Y$ contains a dual edge, then $v$ can have at most three neighboring cells in $Y$, since otherwise $s_4(Y, v)$ wouldn't be a consistent extension and we would be violating Theorem 5.10. Thus (4) will be satisfied by however we define $e$ on this edge. However we define $e$ will satisfy (2), so we need only worry about (5) and uniqueness. Suppose $d_1$ and $d_2$ are dual cells in $Y$ which are adjacent to $v$. Since $Y$ is dually connected to some connected portion of the boundary in $Y$, we know that there $Y$ contains a path $P$ from $d_1$ to $d_2$, possibly going through portions of the boundary. Let $p$ be the single primal vertex in $Y$ which is adjacent to $v$. We can form the consistent simple extension $Y'''$ of $Y$ by applying $s_2$ at every dual edge in $P$. Then $s_3(Y, v)$ would contain all of the dual edges except for one around the face of the dual graph which contains $v$ The dual edge that it would not contain would be the single dual edge in $s_3(Y, v)$. As before, Theorem 5.11 shows that $Y$ must already contain these dual edges. Because of condition (3), $p$ must be an interior cell, then, as we have shown, $Y$ contains all of the dual edges except for one in the dual face containing $v$. We define $e$ on the single edge in $s_4(Y, v) \setminus Y$ so that the line integral of $e$ around the face containing $v$ evaluates to $q(p)f(p)$, and hence $f$ will satisfy 5.

We simply repeat the above processes until we reach $Cl_{\mathcal{E}}(X \cup \{a\})$.

$\square$

Combining Lemma 5.15 and Theorem 5.10, we see there is a mixed problem on the boundary all of whose solutions are zero on one side of a geodesic passing through a boundary spike, and have value 1 on the boundary spike, and that furthermore, this mixed problem always has a solution.

If we have such a situation, then we can solve for $q$ at the boundary vertex $v$ of a boundary spike, since if $\psi$ is the current leaving the boundary spike, and is the voltage at the boundary vertex of the boundary spike, then

$$(1 - 0)q(v) = \psi.$$

We now need to show that if we know $q(v)$ on a boundary spike, then we can find the response matrix for the network with the boundary spike contracted. To do this, fix a boundary potential function $\Phi$ at all of the boundary nodes except $v$ (our boundary spike). Let $u_1$ be the voltage at the interior node of the interior vertex of the boundary spike and let $u_2$ be the voltage at the boundary vertex. Let $\gamma$ be the conductivity along the boundary spike. Then we have

$$(u_2 - u_1)\gamma + u_2 q_v = e_v^T \Lambda_q(\Phi),$$

and hence

$$u_2(\gamma + q_v - (\Lambda_q)_{vv}) = u_1\gamma + K,$$

where $K$ is independent of $u_2$ and $e_v$ is the unit vector with entry 1 in the $v$ component. Thus we can pick $u_1$ (the interior potential) arbitrarily iff $(\gamma + q_v - (\Lambda_q)_v v) \neq 0$. This is easy to show though, since $\gamma + q_v - (\Lambda_q)_v v = 0$ iff the current coming out of node $v$ is $\gamma + q_v$ when we apply the voltage function $e_v$. This would make $u_2 = 0$, but this can't happen, since by the maximum principle, this would imply that the voltage over the entire network was zero except at $v$, but since $u_1 = 0$, and there is current entering that node, there would have to be some other vertex in the graph other than $v$ with nonzero voltage, which is a contradiction. Hence we can pick $u_2$ to force any value of $u_1$, and thus read off the response matrix appropriately.

The above computation shows that we can contract boundary spikes. Since it is trivial to recover $q$ if $G$ consists of only a single boundary vertex, and by the Lemmas at the beginning of the section about contracting and deleting edges of CCP graphs, we know that we can always recover $q$. *QED*

# References


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